P: ISSN No. 0976-8602 RNI No.UPENG/2012/42622 VOL.-7, ISSUE-3, July-2018

E: ISSN No. 2349-9443 Asian Resonance

A New Application of Aboodh Transform for Solving Linear Volterra Integral Equations

Sudhanshu Aggarwal
Assistant Professor,
Deptt.of Mathematics,
National P.G. College,
Barhalganj, Gorakhpur,
U.P., India

Nidhi Sharma
Assistant Professor,
Deptt.of Mathematics,
Noida Institute of Engineering & Technology,
Greater Noida, U.P., India

Raman Chauhan
Assistant Professor,
Deptt.of Mathematics,
Noida Institute of Engineering & Technology,
Greater Noida, U.P., India

In this paper, we used Aboodh transform for solving linear Volterra integral equations and some applications are given in order to demonstrate the effectiveness of Aboodh transform for solving linear Volterra integral equations.

Keywords: Linear Volterra Integral Equation, Aboodh Transform, Convolution Theorem, Inverse Aboodh Transform.

Introduction

Volterra examined the linear Volterra integral equation of the form [1-5]

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \text{ .... } (1)$$

where the unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $k(x,t)$and the function $f(x)$ are given real-valued functions, and λ is a parameter. The Volterra integral equations are the resultant of mathematical modeling of the real life problems. There are a number of process and phenomenon in different area of science and engineering where Volterra integral equations play vital role (like in population growth model, neutron diffusion, biological population, wave propagation etc.).

The Aboodh transform of the function $F(t)$is defined as [6, 8, 9,]:

$$A\{F(t)\} = \frac{1}{v}\int_0^\infty F(t)e^{-vt}dt = K(v), t \geq 0, 0 < k_1 \leq v \leq k_2,$$

where A is Aboodh transform operator.

The Aboodh transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Aboodh transform of the function $F(t)$.

Aboodh [7] gave the application of new transform "Aboodh Transform" to partial differential equations. Aboodh et al. [8] discussed the connection of Aboodh transform with some famous integral transforms. Aboodh et al. [9] solved delay differential equations using Aboodh transformation method. Aboodh et al. [10] gave the solution of ordinary differential equation with variable coefficients using Aboodh transform. Solution of partial integro-differential equations by using Aboodh and double Aboodh transforms methods was given by Aboodh et al [11].

The aim of this work is to establish exact solutions for linear Volterra integral equation using Aboodh transform without large computational work.

Linearity Property of Aboodh Transforms [8]

$A\{aF(t) + bG(t)\} = aA\{F(t)\} + bA\{G(t)\}$**,** where a and b are arbitrary constants.

Aboodh Transform of Some Elementary Functions [6, 9]

S.N.	$F(t)$	$A\{F(t)\} = K(v)$
1.	1	$\frac{1}{v^2}$
2.	t	$\frac{1}{v^3}$
3.	t^2	$\frac{2!}{v^4}$
4.	$t^n, n \geq 0$	$\frac{n!}{v^{n+2}}$
5.	e^{at}	$\frac{1}{v^2 - av}$
6.	$sinat$	$\frac{a}{v(v^2 + a^2)}$
7.	$cosat$	$\frac{1}{v^2 + a^2}$
8.	$sinhat$	$\frac{a}{v(v^2 - a^2)}$
9.	$coshat$	$\frac{1}{v^2 - a^2}$

Aboodh Transform of the Derivatives of the Function $F(t)$ [6, 8]

If $A\{F(t)\} = K(v)$ then

a) $A\{F'(t)\} = vK(v) - \frac{F(0)}{v}$

b) $A\{F''(t)\} = v^2K(v) - \frac{F'(0)}{v} - F(0)$

c) $A\{F^{(n)}(t)\} = v^nK(v) - \frac{F(0)}{v^{2-n}} - \frac{F'(0)}{v^{3-n}} - \cdots \ldots - \frac{F^{(n-1)}(0)}{v}$

Inverse Aboodh Transform

If $A\{F(t)\} = K(v)$ then $F(t)$ is called the inverse Aboodh transform of $K(v)$ and mathematically it is defined as

$$F(t) = A^{-1}\{K(v)\}$$

where A^{-1} is the inverse Aboodh transform operator.

Inverse Aboodh Transform of Some Elementary Functions

S.N.	$K(v)$	$F(t) = A^{-1}\{K(v)\}$
1.	$\frac{1}{v^2}$	1
2.	$\frac{1}{v^3}$	t
3.	$\frac{1}{v^4}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^{n+2}}, n \geq 0$	$\frac{t^n}{n!}$
5.	$\frac{1}{v^2 - av}$	e^{at}
6.	$\frac{1}{v(v^2 + a^2)}$	$\frac{sinat}{a}$
7.	$\frac{1}{v^2 + a^2}$	$cosat$
8.	$\frac{1}{v(v^2 - a^2)}$	$\frac{sinhat}{a}$
9.	$\frac{1}{v^2 - a^2}$	$coshat$

Convolution of Two Functions [12]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx$$
$$= \int_0^t F(t-x)G(x)dx$$

Convolution Theorem for Aboodh Transforms [8]

If $A\{F(t)\} = H(v)$ and $A\{G(t)\} = I(v)$ then

$$A\{F(t) * G(t)\} = vA\{F(t)\}A\{G(t)\} = vH(v)I(v)$$

Aboodh Transforms for Linear Volterra Integral Equations

In this work we will assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed by the difference $(x-t)$. The linear Volterra integral equation (1) can thus be expressed as

$u(x) = f(x) + \lambda \int_0^x k(x-t)u(t)dt$ ……………..(2)

Applying the Aboodh transform to both sides of(2), we have

$A\{u(x)\} = A\{f(x)\} + \lambda A\{\int_0^x k(x-t)u(t)dt\}$…...(3)

Using convolution theorem of Aboodh transform, we have

$A\{u(x)\} = A\{f(x)\} + \lambda vA\{k(x)\}A\{u(x)\}$

$\Rightarrow [1 - \lambda vA\{k(x)\}]A\{u(x)\} = A\{f(x)\}$

$\Rightarrow A\{u(x)\} = \frac{A\{f(x)\}}{[1-\lambda vA\{k(x)\}]}$ … … … … … … … … … ….. . (4)

Operating inverse Aboodh transform on both sides of (4), we have

$u(x) = A^{-1}\left\{\frac{A\{f(x)\}}{[1-\lambda vA\{k(x)\}]}\right\}$……… … … … … … ….. (5)

Which is the required solution of (2).

Applications

In this section, some applications are given in order to demonstrate the effectiveness of Aboodh transform for solving linear Volterra integral equations.

Application: 1

Consider linear Volterra integral equation with $\lambda = -1$

$$u(x) = 1 - \int_0^x (x-t)\,u(t)dt\text{…………(6)}$$

Applying the Aboodh transform to both sides of(6), we have

$A\{u(x)\} = \frac{1}{v^2} - A\{\int_0^x (x-t)\,u(t)dt\}$……………. (7)

Using convolution theorem of Aboodh transform on (7), we have

$A\{u(x)\} = \frac{1}{1+v^2}$………… (8)

Operating inverse Aboodh transform on both sides of(8), we have

$u(x) = A^{-1}\left\{\frac{1}{1+v^2}\right\} = cosx$……………. … … …. . (9)

which is the required exact solution of (6).

Application: 2

Consider linear Volterra integral equation with $\lambda = -1$

$$u(x) = cosx + sinx - \int_0^x u(t)\,dt\text{…… (10)}$$

Applying the Aboodh transform to both sides of (10), we have

$A\{u(x)\} = \frac{1}{1+v^2} + \frac{1}{v(1+v^2)} - A\{\int_0^x u(t)\,dt\}$……. (11)

Using convolution theorem of Aboodh transform on(11), we have

$A\{u(x)\} = \frac{1}{1+v^2}$………… … … … … … … … … … … (12)

Operating inverse Aboodh transform on both sides of(12), we have

$u(x) = A^{-1}\left\{\frac{1}{1+v^2}\right\} = cosx$……………..… … …..(13)

which is the required exact solution of (10).

Application: 3

Consider linear Volterra integral equation with $\lambda = 1$

$$u(x) = 1 - x + \int_0^x (x-t)u(t)\,dt\ \text{……}\ (14)$$

Applying the Aboodh transform to both sides of(14), we have

$A\{u(x)\} = \frac{1}{v^2} - \frac{1}{v^3} + A\{\int_0^x (x-t)u(t)\,dt\}$…. …..(15)

Using convolution theorem of Aboodh transform on(15), we have

$A\{u(x)\} = \frac{1}{v^2+v}$…………………………………. (16)

Operating inverse Aboodh transform on both sides of(16), we have

$u(x) = A^{-1}\left\{\frac{1}{v^2+v}\right\} = e^{-x}$ ………………………. (17)

which is the required exact solution of (14).

Application: 4

Consider linear Volterra integral equation with $\lambda = -1$

$$u(x) = x - \int_0^x (x-t)u(t)\,dt\ \text{…………}\ (18)$$

Applying the Aboodh transform to both sides of(18), we have

$A\{u(x)\} = \frac{1}{v^3} - A\{\int_0^x (x-t)u(t)\,dt\}$…. … … … ….(19)

Using convolution theorem of Aboodh transform on(19), we have

$A\{u(x)\} = \frac{1}{v(1+v^2)}$…………………………………. (20)

Operating inverse Aboodh transform on both sides of(20), we have

$u(x) = A^{-1}\left\{\frac{1}{v(1+v^2)}\right\} = sinx$………… … … … … …(21)

which is the required exact solution of (18).

Application: 5

Consider linear Volterra integral equation with $\lambda = 1$

$$u(x) = 1 - \frac{x^2}{2} + \int_0^x u(t)\,dt\ \text{………………}(22)$$

Applying the Aboodh transform to both sides of (22), we have

$A\{u(x)\} = \frac{1}{v^2} - \frac{1}{v^4} + A\{\int_0^x u(t)\,dt\}$……………….. (23)

Using convolution theorem of Aboodh transform on(23), we have

$A\{u(x)\} = \frac{v-1}{v^3} = \frac{1}{v^2} - \frac{1}{v^3}$……………… … … … … … … (24)

Operating inverse Aboodh transform on both sides of(16), we have

$$u(x) = A^{-1}\left\{\frac{1}{v^2}\right\} - A^{-1}\left\{\frac{1}{v^3}\right\}$$
$$= 1 + x\text{…………… … … … … … … … … … … ….}(25)$$

which is the required exact solution of (22).

Conclusion

In this paper, we have successfully developed the Aboodh transform for solving linear Volterra integral equations. The given applications showed that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear Volterra integral equations and their system.

References

Aboodh, K.S. (2013) The new integral transform "Aboodh Transform", Global Journal of Pure and Applied Mathematics, 9(1), 35-43.

Aboodh, K.S. (2014) Application of new transform "Aboodh Transform" to partial differential equations, Global Journal of Pure and Applied Mathematics, 10(2), 249-254.

Aboodh, K.S., Farah, R.A., Almardy, I.A. and Almostafa, F.A. (2016) Solution of partial integro-differential equations by using Aboodh and double Aboodh transforms methods, Global Journal of Pure and Applied Mathematics, 13(8), 4347-4360.

Aboodh, K.S., Farah, R.A., Almardy, I.A. and Osman, A.K. (2018) Solving delay differential equations by Aboodh transformation method, International Journal of Applied Mathematics & Statistical Sciences, 7(2), 55-64.

Aboodh, K.S., Idris Abdullahi and Nuruddeen, R.I. (2017) On the Aboodh transform connections with some famous integral transforms, International Journal of Engineering and Information Systems, 1(9), 143-151.

Jerri, A. J. (1999) Introduction to integral equations with applications, Wiley, New York.

Kanwal, R. (1997) Linear integral equations, Birkhauser, Berlin.

Linz, P. (1984) Analytical and numerical methods for Volterra equations, SIAM, Philadelphia.

Lokenath Debnath and Bhatta, D.(2007) Integral transforms and their applications, Second edition, Chapman & Hall/ CRC.

Mohand, D., Aboodh, K.S. and Abdelbagy, A. (2016) On the solution of ordinary differential equation with variable coefficients using Aboodh transform, Advances in Theoretical and Applied Mathematics, 11(4), 383-389.

Rahman, M. (2007) Integral equations and their application, with press, Southampton, Boston.

Wazwaz, A.M. (1997) A first course in integral equations, World scientific, Singapore.